There are <u>three</u> questions, with point values as marked (100 pt. total).
Time allowed: 80 minutes

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1. *Geostrophic Adjustment* [30 points]

As in class, we will consider geostrophic adjustment for quasi-geostrophic, f-plane flow in the shallow water equations. There is no basic state flow, i.e., in the initial state, the fluid is at rest ($\overline{u} = \overline{v} = 0$). We perturb the fluid here by adding an initial meridional wind

$$v' = \begin{cases} v_o & x > L \\ (v_o x/L) & -L < x < L \\ -v_o & x < -L \end{cases}$$, while keeping h' = 0.

(a) Is this flow/height configuration in geostrophic balance? [Hint: No] Explain your answer. [Hint: What forces balance in geostrophic balance?]

Geostrophic balance requires a balance between a pressure-gradient force and a Coriolis force. It is not in geostrophic balance because fv' produces a Coriolis force, but there is no pressure-gradient force to balance it, since $h' = 0$, so that $\nabla h' = 0$.

(b) What is the *potential* vorticity at the initial time?

For potential vorticity, $Q' = \dfrac{\zeta}{f} - \dfrac{h'}{H}$. We know that at the initial time, $h' = 0$. However, the initial vorticity is not zero. In the region $-L < x < L$, $\zeta = \dfrac{\partial v'}{\partial x} - \dfrac{\partial u'}{\partial y} = \dfrac{v_o}{L}$, so that $Q'(t=0) = \dfrac{v_o}{fL}$.

Note that the numerator is v_o, the letter v, and not the Greek letter nu, ν.

(c) What is the *relative* vorticity at any later time in terms of h' and constant parameters?

We need to use the fact that potential vorticity does not change with time, though the contributors to it might change. Thus $Q'(t=0) = \dfrac{v_o}{fL} = Q'(t>0) = \dfrac{\zeta(t>0)}{f} - \dfrac{h'(t>0)}{H}$.

Rearranging this gives $\zeta(t>0) = \dfrac{v_o}{L} + \dfrac{fh'(t>0)}{H}$.

2. *Inertio-gravity waves* [50 points]

If we assume that gravity waves can oscillate slowly, then Coriolis effects must be included in their description. On an f-plane with no basic state flow, the perturbation equations are

$$\frac{\partial u'}{\partial t} = -g\frac{\partial h'}{\partial x} + fv' \quad (1)$$

$$\frac{\partial v'}{\partial t} = -g\frac{\partial h'}{\partial y} - fu' \quad (2)$$

$$\frac{\partial h'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \qquad (3)$$

where u', v', and h' (zonal wind, meridional wind and height perturbations) are unknowns.

(a) Why do we need momentum equations in both the x and y directions?
We need momentum equations in both directions because motion in one direction will produce acceleration and thus motion in the other, through the Coriolis force.

(b) One can combine (1), (2) and (3) and use conservation of potential vorticity to show that

$$\frac{\partial^2 h'}{\partial t^2} - gH\left(\nabla^2 h'\right) + f^2 h' = 0 \qquad (4)$$

Assume $h' = h_o \exp\{i(kx + ly - \nu t)\}$, where ν is frequency (try not to confuse frequency ν and meridional wind v), k is zonal wavenumber and *l* is meridional wavenumber. Show that the dispersion relation for inertio-gravity waves in the shallow water equations is

$$\nu^2 = f^2 + gH\left(k^2 + l^2\right)$$

We have to substitute the assumed wave form for h′ into (4). Recognize that from past problem sets and class lectures,

$$\frac{\partial h'}{\partial t} = -i\nu h'$$

$$\frac{\partial h'}{\partial x} = +ikh'$$

$$\frac{\partial h'}{\partial y} = +ilh'$$

$$\nabla^2 h' = \frac{\partial^2}{\partial x^2}h' + \frac{\partial^2}{\partial y^2}h' = (ik)^2 h' + (il)^2 h' = -\left(k^2 + l^2\right)h'$$

Thus, substituting into (4) gives

$$\left\{-(\nu)^2 + gH\left(k^2 + l^2\right) + f^2\right\}h' = 0$$

for which the non-trivial result (h′ ≠ 0) is $\nu^2 = f^2 + gH\left(k^2 + l^2\right)$.

(c) What is the phase speed in the x and y directions for these waves?

$C_x = \nu/k = \pm\sqrt{f^2 + gH\left(k^2 + l^2\right)}\Big/k$; $C_y = \nu/l = \pm\sqrt{f^2 + gH\left(k^2 + l^2\right)}\Big/l$

(d) What is the group *velocity* for these waves?

$\vec{c}_g = \left(\vec{c}_g\right)_x \hat{i} + \left(\vec{c}_g\right)_y \hat{j}$ where

$$\left(\vec{c}_g\right)_x = \frac{\partial \nu}{\partial k} = \pm\frac{\tfrac{1}{2}}{\sqrt{f^2 + gH\left(k^2 + l^2\right)}}\left\{2gHk\right\} = \pm\frac{gHk}{\sqrt{f^2 + gH\left(k^2 + l^2\right)}}$$

and

$$\left(\vec{c}_g\right)_y = \frac{\partial \nu}{\partial l} = \pm \frac{1/2}{\sqrt{f^2 + gH\left(k^2 + l^2\right)}}\left\{2gHl\right\} = \pm \frac{gHl}{\sqrt{f^2 + gH\left(k^2 + l^2\right)}} \ .$$

(e) For what values of f and meridional wavenumber *l* does the component of group velocity in the x direction equal the phase speed in the x direction?

Need $\left(\vec{c}_g\right)_x = \pm \frac{gHk}{\sqrt{f^2 + gH\left(k^2 + l^2\right)}}$ $= C_x =$ $\pm\sqrt{f^2 + gH\left(k^2 + l^2\right)}\Big/k$, or

$$\pm \frac{gHk}{\sqrt{f^2 + gH\left(k^2 + l^2\right)}} = \pm \frac{\sqrt{f^2 + gH\left(k^2 + l^2\right)}}{k}$$

or

$$gHk^2 = f^2 + gH\left(k^2 + l^2\right)$$

so that

$$f^2 + gHl^2 = 0$$

Since f and *l* (note that this is the letter, *l*) must both be real and not imaginary, then we can only have f = *l* = 0 .

3. *Rossby waves* [20 points]

The figure below shows a distribution of 500 mb geopotential height. There are two Rossby waves present. At 45 N, their zonal wavelengths are

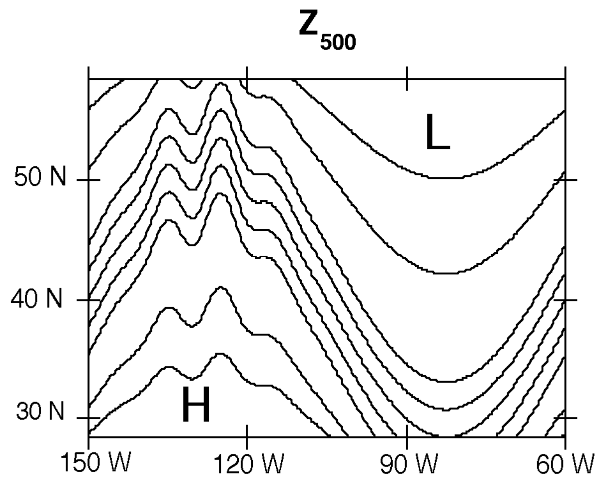
$L_1 = 5000$ km

$L_2 = \ 600$ km

At 45 N, there is also a zonal flow at 500 mb,

$U = +5$ m/s

and $\beta = 1.611 \cdot 10^{-11} m^{-1} s^{-1}$



(a) Assume that

i. the behavior of these waves is well-approximated by a barotropic-flow model

ii. the meridional wave number for both waves is $l = 0$.

What is the phase speed c_x for both waves? (*Be careful with your units!*)

$C_x = \overline{U} - \dfrac{\beta}{\left(k^2 + l^2\right)} = \overline{U} - \dfrac{\beta}{k^2}$, since $l = 0$. We need $k = 2\pi/L$

$k_1 = 2\pi/L_1 = 2\pi/5 \cdot 10^6$ m $= 1.26 \cdot 10^{-6}$ m^{-1}

$k_2 = 2\pi/L_2 = 2\pi/6 \cdot 10^5$ m $= 1.05 \cdot 10^{-5}$ m^{-1}

Then

$C_{x1} = 5$ m/s - $1.611.10^{-11}$ m^{-1} s^{-1}/$(1.26 \cdot 10^{-6}$ m$^{-1})^2$ = -5.15 m/s

$C_{x2} = 5$ m/s - $1.611.10^{-11}$ m^{-1} s^{-1}/$(1.05 \cdot 10^{-5}$ m$^{-1})^2$ = +4.85 m/s

(b) Suppose U(500 mb) was something other than +5 m/s. What would it have to be to get $c_x = 0$ for the long wave, L_1?

We would need $\overline{U} - \dfrac{\beta}{k^2} = 0$, or $\overline{U} = \dfrac{\beta}{k^2}$ = $1.611.10^{-11}$ m^{-1} s^{-1}/$(1.26 \cdot 10^{-6}$ m$^{-1})^2$ = +10.15 m/s

Equations That Might Be Useful

$$\frac{\partial^2}{\partial t^2}h' - c^2\left(\frac{\partial^2}{\partial x^2}h' + \frac{\partial^2}{\partial y^2}h'\right) + f_o H\zeta' = 0$$

$$\frac{\partial}{\partial t}p_s = -\int_0^1 \nabla\cdot\left(p_s\vec{V}\right)d\sigma$$

$$\vec{V}_\psi = \hat{k}\times\vec{\nabla}\psi$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$u = -\frac{\partial\psi}{\partial y}$$

$$v = +\frac{\partial\psi}{\partial x}$$

$$c = U - \frac{\beta}{K^2}$$

$$\lambda_R = \frac{\sqrt{gH}}{f_o}$$

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) = -\vec{V}_\psi\cdot\nabla\left(\nabla^2\psi + f\right)$$

$$\iint_{A_\psi}\vec{V}_\psi\cdot\nabla\left(\nabla^2\psi + f\right)dA_\psi = 0$$

$$c = \begin{pmatrix}\nu/k\\ \nu/l\\ \nu/m\end{pmatrix}$$

$$\vec{c}_g = \begin{pmatrix}\partial\nu/\partial k\\ \partial\nu/\partial l\\ \partial\nu/\partial m\end{pmatrix}$$

$$Q'\left(x,y,t\right) = \zeta'/f_o - h'/H = const$$

$$\frac{\partial}{\partial t}u' - f_o v' = -g\frac{\partial}{\partial x}h'$$

$$\frac{\partial}{\partial t}v' + f_o u' = -g\frac{\partial}{\partial y}h'$$

$$\frac{\partial}{\partial t}h' + H\left(\frac{\partial}{\partial x}u' + \frac{\partial}{\partial y}v'\right) = 0$$