There are <u>three</u> questions, with point values as marked (100 pt. total).
Time allowed: exam period

1. *Instability with friction* [40 points]

Suppose we have frictional drag present in our two-layer model. One way to represent friction per unit mass is to assume it is given by Ekman pumping from the top and bottom boundaries. Then the level-1 and level-3 vorticity equations in our two-layer model can be written

$$\frac{\partial}{\partial t}\nabla^2\psi_1 + \vec{v}_1\cdot\nabla\left\{\nabla^2\psi_1 + f\right\} = -\varepsilon\nabla^2\psi_1 + \frac{f_o}{\Delta p}\omega_2 \quad \text{, and} \qquad \text{(Eq. 1)}$$

$$\frac{\partial}{\partial t}\nabla^2\psi_3 + \vec{v}_3\cdot\nabla\left\{\nabla^2\psi_3 + f\right\} = -\varepsilon\nabla^2\psi_3 - \frac{f_o}{\Delta p}\omega_2 \quad , \qquad \text{(Eq. 2)}$$

where ε is a damping constant with units 1/(time). Note that here, the Coriolis parameter is a constant, f_o.

(a) For the basic state we used in class for baroclinic instability, $\Psi_j = -U_j y$ and $\omega_j = 0$, derive the linearized ***level-1*** vorticity equation for perturbation vorticity, $\nabla^2\psi_1'$. Again, f_o is a constant.

The friction enters our computations through the terms involving ε.
We need to use the first equation above. First note that for the basic state we have

$$\nabla^2\Psi_1 = \frac{\partial^2}{\partial x^2}\Psi_1 + \frac{\partial^2}{\partial y^2}\Psi_1 = \frac{\partial^2}{\partial x^2}\left(-U_1 y\right) + \frac{\partial^2}{\partial y^2}\left(-U_1 y\right) = 0 + \frac{\partial}{\partial y}\left(-U_1\right) = 0$$

since U_1 is a constant. Also we have

$$\vec{V}_1 = \left(U_1 + u_1'\right)\hat{x} + \left(v_1'\right)\hat{y} = \left(U_1 - \frac{\partial}{\partial y}\psi_1'\right)\hat{x} + \left(\frac{\partial}{\partial x}\psi_1'\right)\hat{y}$$

Then substituting the breakdown $\psi_1 = \Psi_1 + \psi_1' \quad \omega_2 = \omega_2'$

$$\frac{\partial}{\partial t}\nabla^2\left(\Psi_1 + \psi_1'\right) + \vec{v}_1\cdot\nabla\left\{\nabla^2\Psi_1 + \nabla^2\psi_1' + f\right\} = -\varepsilon\nabla^2\Psi_1 - \varepsilon\nabla^2\psi_1' + \frac{f_o}{\Delta p}\omega_2'$$

or, using the information above,

$$\frac{\partial}{\partial t}\nabla^2\left(\psi_1'\right) + \left[\left(U_1 - \frac{\partial}{\partial y}\psi_1'\right)\hat{x} + \left(\frac{\partial}{\partial x}\psi_1'\right)\hat{y}\right]\cdot\nabla\left\{\nabla^2\psi_1' + f\right\} = -\varepsilon\nabla^2\psi_1' + \frac{f_o}{\Delta p}\omega_2'$$

This gives, recognizing that f is a function of y and not of x and using $\beta = df/dy$

$$\frac{\partial}{\partial t}\nabla^2\left(\psi_1'\right) + U_1\frac{\partial}{\partial x}\nabla^2\psi_1' + \left(\frac{\partial}{\partial x}\psi_1'\right)\beta = -\varepsilon\nabla^2\psi_1' + \frac{f_o}{\Delta p}\omega_2'$$

A necessary condition for this to be the correct final result is that each term in the equation contains one and only one perturbation quantity. This condition by itself does not guarantee that the result is correct, but if it was not satisfied, it would guarantee that the result is wrong.

(b) For the f-plane ($\beta = 0$) with perturbations all proportional to $\exp\{ik(x-ct)\}$, the dispersion relation is

$$c = U_m - \frac{i\varepsilon\left(k^2+\lambda^2\right)}{k\left(k^2+2\lambda^2\right)} \pm \sqrt{\delta_\varepsilon}\ , \qquad \text{(Eq. 3)}$$

where

$$\delta_\varepsilon = \frac{U_T^2\left(k^2-2\lambda^2\right)}{\left(k^2+2\lambda^2\right)} - \frac{\varepsilon^2\lambda^4}{k^2\left(k^2+2\lambda^2\right)^2}\ .$$

Suppose $U_T = 0$, what is c?

If $U_T = 0$, then

$$\delta_\varepsilon = -\frac{\varepsilon^2\lambda^4}{k^2\left(k^2+2\lambda^2\right)^2}$$, and so

$$c = U_m - \frac{i\varepsilon\left(k^2+\lambda^2\right)}{k\left(k^2+2\lambda^2\right)} \pm \frac{i\varepsilon\lambda^2}{k\left(k^2+2\lambda^2\right)}$$

Adding and subtracting according to the ±, we have

$$c = U_m - \frac{i\varepsilon}{k} \qquad c = U_m - \frac{i\varepsilon k}{\left(k^2+2\lambda^2\right)}$$

as the two answers.

(c) Looking carefully at the dispersion relation above, with $U_T \neq 0$, if the zonal wavenumber k is large enough, then $\delta_\varepsilon > 0$, but Im(c) is still nonzero. Is the perturbation in this case unstable - that is, is it exponentially *growing*? Be sure to show why you arrive at your answer and why this effect of friction makes physical sense.

We need to recall that for this problem,

$$\psi_1' = \tilde{A}\exp\{ik(x - ct)\} = \tilde{A}\exp\{ik(x - \mathrm{Re}(c)t - i\,\mathrm{Im}(c)t)\}$$

or,

$$\psi_1' = \tilde{A}\exp\{ik(x - \mathrm{Re}(c)t)\}\exp\{k\,\mathrm{Im}(c)t\} \qquad \text{(Eq. 4)}$$

where c has a real and imaginary part, c = Re(c) + *i**Im(c) .

When $\delta_\varepsilon > 0$, the only term in the dispersion relationship, (Eq. 3), that is multiplied by "*i*" is the second one, so that

$$\mathrm{Im}(c) = -\frac{\varepsilon(k^2 + \lambda^2)}{k(k^2 + 2\lambda^2)} \qquad \text{(Eq. 5)}$$

(Note that the imaginary part is everything that is multiplied by "*i*", but does not include "*i*" itself.)

The key factor in (Eq. 5) is that Im(c) < 0, since our wavenumber k is positive, and ε must be positive, too. (If ε > 0 , then friction would be causing the flow to speed up, which is physically wrong.) Since Im(c) < 0, then the term $\exp\{k\,\mathrm{Im}(c)t\}$ in (Eq. 4) is exponentially ***decaying***. Thus, even though Im(c) ≠ 0, it is not producing exponential growth, but decay. Of course, this is what we expect the ε term by itself to do: friction decays the flow.

(d) The phase speed c here is really just the phase speed in the x direction, c_x, because the only nonzero wavenumber in this problem is k (i.e., $l = m = 0$). Assume that $k^2 >> \lambda^2$ (i.e., the waves are very, very short), so that $\frac{(k^2 - 2\lambda^2)}{(k^2 + 2\lambda^2)} \approx \frac{(k^2)}{(k^2)} = 1$. Assume also that ε = 0 (no friction). What then is the *group velocity* of this wave in the x direction? It helps to write first what c becomes under these two assumptions.

Under these assumptions,

$$\delta_\varepsilon = \frac{U_T^2(k^2 - 2\lambda^2)}{(k^2 + 2\lambda^2)} \approx U_T^2$$

so that (Eq. 3) becomes

$$c = U_m \pm U_T$$

This is a non-dispersive relationship, because phase speed does not depend on wavenumber (or wavelength). In fact, the group velocity in the x direction is the same as the phase speed when the wave is non-dispersive:

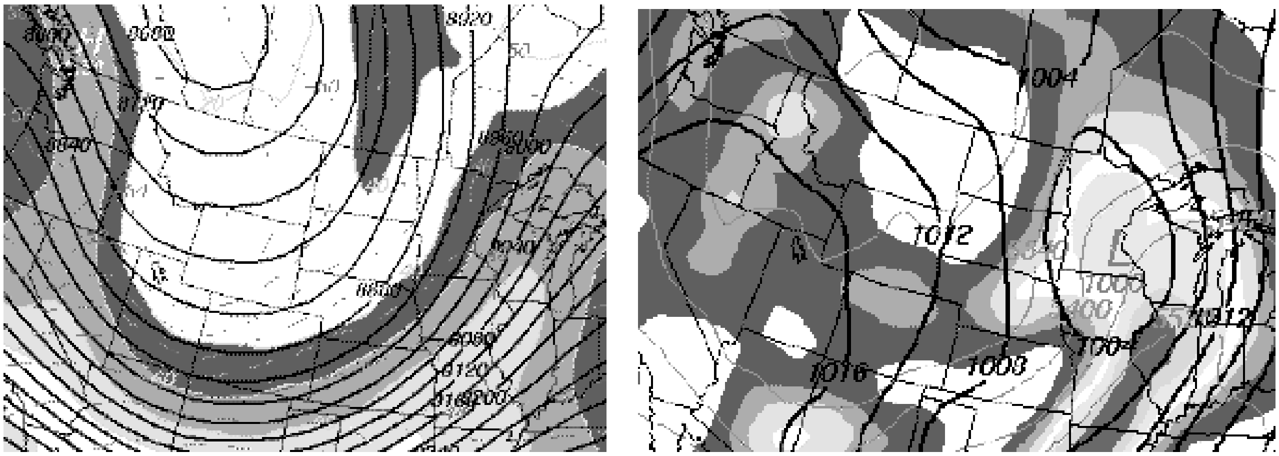
$$(\vec{c}_g)_x = \frac{\partial \nu}{\partial k} = \frac{\partial}{\partial k}(kc) = c\frac{\partial}{\partial k}k = c$$

since c does not vary with k.

2. *Synoptic development* [30 points]

The figures below show forecasts of - LEFT: 300 hPa heights (m; heavy contours), and RIGHT: mean sea-level pressure (hPa; heavy contours) and 1000-500 hPa thickness (m; light contours)

Pressure contours are every 4 hPa, thickness contours are every 60 m, and height contours are every 40 m. You can ignore the gray shades for now.



(a) Using the positions of surface pressure and 300 hPa height contours and thinking in terms of baroclinically unstable waves, this configuation describes a system that is
(i) growing, (ii) decaying, (iii) unchanging, or (iv) none of the above.

It is *growing*, answer (i), because the upper level trough lies to the west of the surface low. In other words, the system is tilted to the west with increasing height.

(b) The gray shading in the right panel indicates forecast precipitation amounts. The precipitation is produced by the vertical motions associated with the storm. Wisconsin and Illinois have moderate amounts forecasted, while North and South Dakota have very little or none. This precipitation distribution tells us that
(i) there is downward motion over North and South Dakota
(ii) there is upward motion over Wisconsin and Illinois
(iii) both (i) and (ii) are occurring
(iv) there is upward motion over North and South Dakota

There is upward motion over Wisconsin and Illinois, because upward motion leads to condensation and thus precipitation. There is most likely downward motion over North and South Dakota, because downward motion suppresses rainfall. This pattern of upward/downward motion is consistent with a growing baroclinic wave, which will have upward motion ahead of the surface low and sinking motion behind it.

So the answer is (iii), both (i) and (ii) are occurring.

3. *Longwave cutoff* [30 points]
An instability theorem states that a baroclinically unstable wave must propagate at a speed between the minimum and maximum wind speeds in the atmosphere. In our two-layer model, we must then have

$$\bar{u}_3 < \mathrm{Re}\{c\} < \bar{u}_1 \quad , \qquad (3.1)$$

where c is the general, complex phase speed (i.e., c = Re{c] + i *Im{c}, where Im{c} is the imaginary part of c (part of c multiplied by i) and Re{c} is the part of c not multiplied by i).

(a) Consider our two-layer model's dispersion relationship for c in the general case ($U_T \neq 0$, $\beta \neq 0$). In this case,

$$c = U_m - \frac{\beta\left(k^2+\lambda^2\right)}{k^2\left(k^2+2\lambda^2\right)} \pm \sqrt{\delta} \quad ,$$

where $\delta = \dfrac{\beta^2\lambda^4}{k^4\left(k^2+2\lambda^2\right)^2} - U_T^2\dfrac{2\lambda^2-k^2}{2\lambda^2+k^2}$ and $U_m = \frac{1}{2}\left(U_1+U_3\right)$

For an *unstable* wave, show that the real part of c, Re{c) is

$$\mathrm{Re}\{c\} = U_m - \frac{\beta\left(k^2+\lambda^2\right)}{k^2\left(k^2+2\lambda^2\right)} \qquad (3.2)$$

For an unstable wave, we need Im(c) ≠ 0, and the only way that can happen is if $\delta < 0$, making the square root of δ a term multiplied by *i*. Then the part of c not multiplied by *i*, is the rest of c, as given by (3.2).

(b) For waves with long wavelength, the wavenumber k becomes small. Using (3.2), show that for very long waves, where $k^2 \ll \lambda^2$, we then have

$$\mathrm{Re}\{c\} \approx U_m - \beta/2k^2 \quad , \qquad (3.3)$$

In (3.2), if k becomes very small, then

$$\mathrm{Re}\{c\} = U_m - \frac{\beta\left(k^2+\lambda^2\right)}{k^2\left(k^2+2\lambda^2\right)} \approx U_m - \frac{\beta\left(\lambda^2\right)}{k^2\left(2\lambda^2\right)} = U_m - \frac{\beta}{2k^2}$$

(c) Using (3.1) and (3.3), explain why a wave will be stable if its wavelength is very long.

We need to have Re{c} lie between the wind speeds in the upper and lower levels. However, if k becomes very small, then

$$\mathrm{Re}\{c\} \approx -\frac{\beta}{2k^2}$$

This quantity will become a large, negative number as k gets smaller, so that at some point as k shrinks, Re{c} will become smaller than the basic state's zonal wind at level 3, that is

$$\mathrm{Re}\{c\} \approx -\frac{\beta}{2k^2} < \bar{u}_3$$

and the condition (3.2) is violated, so the wave must be stable.

(d) If instead we have $\beta = 0$ (i.e., an f-plane), using again (3.1) and (3.3) for this case, explain why very long waves remain unstable, that is, why waves do *not* become stable as k^2 becomes much smaller than λ^2.

If $\beta = 0$, then,

$$\delta = -U_T^2 \frac{2\lambda^2 - k^2}{2\lambda^2 + k^2} \approx -U_T^2 \frac{2\lambda^2}{2\lambda^2} = -U_T^2$$

for waves with very long wavelength (i.e., k very small). Then $\delta < 0$, making the square root of δ a term multiplied by *i*, so that

$$\mathrm{Re}\{c\} = U_m$$

for very long waves. Here, Re{c} does not change as k gets smaller, in contrast to the $\beta \neq 0$ result. Because Re{c} is the average of U_1 and U_3 here, it can never become smaller than U_3, so the waves will remain unstable.

Equations That Might Be Useful

$$\zeta = \frac{\partial \mathrm{v}}{\partial \mathrm{x}} - \frac{\partial \mathrm{u}}{\partial \mathrm{y}} = \nabla^2\psi = \frac{\partial^2\psi}{\partial \mathrm{x}^2} + \frac{\partial^2\psi}{\partial \mathrm{y}^2} \qquad u = -\frac{\partial\psi}{\partial y} \qquad \mathrm{v} = +\frac{\partial\psi}{\partial x}$$

$$\nu = kc_x \qquad c = \begin{pmatrix} \nu/k \\ \nu/l \\ \nu/m \end{pmatrix} \qquad \vec{c}_g = \begin{pmatrix} \partial\nu/\partial k \\ \partial\nu/\partial l \\ \partial\nu/\partial m \end{pmatrix}$$

$$\lambda^2 = \frac{f_o^2}{\sigma(\Delta p)^2} \qquad \omega_2' \approx \mathrm{v}_2' \frac{\partial \bar{T}}{\partial y}$$